MATH-SHU 350

Recitation 2: Probability Space

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Exercise 1. *Let* $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ *be a sequence of probability space. Verify that*

$$\mathcal{G} = \{E_1 \times E_2 \times E_3 \cdots E_n : E_1 \in \mathcal{F}_1, E_2 \in \mathcal{F}_2 \cdots E_n \in \mathcal{F}_n\},$$

is a π*-system.*

Exercise 2. *(Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice? (Suppose that you prefer a car than a goat.)*

Exercise 3. *Let* $\Omega = \{1, 2, 3, 4\}$*,* $\mathcal{F}$ *be the power set of* Ω *and* $\mathbb{P}$ *the probability measure such that*

$$\mathbb{P}[\{1\}] = \mathbb{P}[\{2\}] = \mathbb{P}[\{3\}] = \mathbb{P}[\{4\}].$$

1. *Show that the following two classes of events*

$$\mathcal{C}_1 = \{\{1, 2\}\}, \quad \mathcal{C}_2 = \{\{2, 3\}, \{2, 4\}\},$$

are independent but $\sigma(\mathcal{C}_1)$ *and* $\sigma(\mathcal{C}_2)$ *are not.*

2. *We learned a theorem that establishes an extra condition in which we can conclude* $\sigma(\mathcal{C}_1)$ *and* $\sigma(\mathcal{C}_2)$ *are independent. Identify the condition that is missing in this example* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*.*

Exercise 4. *Let* Ω *be the set of outcome space and* $(X_n)_{n \geqslant 1}$ *a sequence of real-valued random variables defined on* Ω*.*

1. *Describe the following expressions without using the words "for all/any" nor "there exists"*

$$A = \bigcup_{a \in \mathbb{N}} \bigcup_{b \in \mathbb{N}} \bigcap_{n \geqslant 1} \{\omega \in \Omega\,,\ a \leqslant X_n(\omega) \leqslant b\};$$
$$B = \bigcup_{N \geqslant 1} \bigcap_{n \geqslant N} \bigcap_{m \geqslant n} \{\omega \in \Omega\,,\ X_n(\omega) - X_m(\omega) \geqslant 0\};$$
$$C = \bigcup_{k \in \mathbb{N}} \bigcap_{N \geqslant 1} \bigcup_{n \geqslant N} \bigcup_{m \geqslant N} \left\{\omega \in \Omega\,,\ |X_n(\omega) - X_m(\omega)| > \frac{1}{k}\right\}.$$

2. *Do the reverse operation to translate the following events in* Ω

	set of $\omega \in \Omega$ *such that the sequence* $(X_n(\omega))_{n \geqslant 1}$*...*
D	*... is not bounded from above,*
E	*... goes to* $+\infty$ *.*

Exercise 5. *Prove the strong law of large number under the condition of* L^4 *moment.*